QNo> Prove the main theorem.

Proof:- Let $\epsilon > 0$ be given. ~~Hence~~ Then by the above lemma the region within C can be divided into a finite number n of meshes either square or partial squares such that a point z_i exists in each mesh for which

① i.e. $\left| \dfrac{f(z) - f(z_i)}{z - z_i} - f'(z_i) \right| < \epsilon$ is true.

Let $\eta_i(z)$ where $|\eta_i(z)| < \epsilon$.

$\therefore f(z) = f(z_i) - z_i f'(z_i) + z f'(z_i) + \eta_i(z)(z - z_i)$ — Ⓐ

The value of $f(z)$ at any point z on the boundary ~~S_i~~ will given by ~~$f(z) =$~~ Ⓐ therefore If S_i be points, then we have

$$\int_{S_i} f(z)\,dz = [f(z_i) - z_i f'(z_i)] \int_{S_i} dz + f'(z_i) \int_{S_i} z\,dz$$

$$+ \int_{S_i} (z - z_i)\,\eta_i(z)\,dz.$$

$$\int_{S_i} dz = 0, \int_{S_i} z\,dz = 0$$

$$\therefore \int_{S_i} (z - z_i)\,\eta_i(z)\,dz.$$

It is clear ~~for~~ from the given

diagram the sum of the integrals around the closed curve C, since the integral along the common boundary line of pair of adjacent meshes cancel each other only the integrals along the arcs which form parts of C remain,

$$\int_C f(z)\,dz = \sum_{i=1}^{n} \int_{S_i} f(z)\,dz$$

$$= \sum_{i=1}^{n} \int_{S_i} (z - z_i)\,\eta_i(z)\,dz$$

$$\therefore \left|\int_C f(z)\,dz\right| = \sum_{i=1}^{n} \int_{S_i} (z - z_i)\,\eta_i(z)\,dz$$

$$\leq \sum_{i=1}^{n} \int_{S_i} |z - z_i|\,|\eta_i(z)|\,|dz|$$

$$< \epsilon \sum_{i=1}^{n} \int_{S_i} |z - z_i|\,|dz| \quad \text{--- (8)}$$

The boundary S_i of a mesh either completely or partially coincides with the boundary of a square. We let a_i be the length of sides of that square, since z lies on S_i & z_i lies ~~either~~ within S_i the distance between z & z_i cannot exceed the length of diagonal of square $a_i\sqrt{2}$

$$\therefore |z - z_i| \leq a_i\sqrt{2}$$

$\therefore \int_{S_i} |z - z_i| |dz| \leq a_i \sqrt{2} \int_{S_i} |dz|$ — (C)

Now, $\int_{S_i} |dz|$ represent lenth of S_i it will $4a_i$ is a Complete square and it cannot exceed $(4a_i + l_i)$ if S_i is a partial square where l_i is the length of arc of C which forms a part of S_i therefore S_i is complex square,

$$\int_{S_i} |z - z_i| |dz| \leq a_i \sqrt{2}, 4a_i = 4\sqrt{2} a_i^2 \text{ — (D)}$$

And if partial square then

$$\int_{S_i} |z - z_i| |dz| \leq a_i \sqrt{2} (4a_i + l_i) \leq 4\sqrt{2} a_i^2 + \sqrt{2} a l_i \text{ — (E)}$$

Where a is length of side of some square enclosing the entire curve C, together with all the squares used originally in covering C. It is clear that the sum of the areas a_i^2 of all those squares cannot exceed a^2. Hence, if l denotes the length of C, then (B), (D) & (E) give

$$\left| \int_C f(z) \right| dz \leq \epsilon \sum_{i=1}^{n} 4\sqrt{2} a_i^2 + \sqrt{2} a l_i)$$

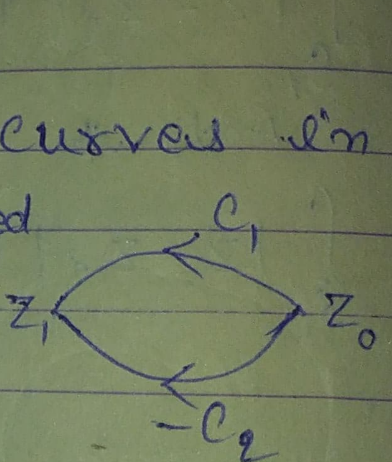
$$\leq \epsilon (4\sqrt{2} a^2 + \sqrt{2} a l)$$

$$= \epsilon K \text{, where K is a Constt.}$$

Since ϵ is arbitrary, it follows that $\int_C f(z)\,dz = 0$.

Cor. 1:- Let $f(z)$ be analytic in a simply connected domain D. Then the integral along every rectifiable curve in D joining any two given points of D is the same i.e. it does not depend on the curve joining the two points.

Proof:- Let C_1, C_2 be any two curves in z_0 & z_1. Let C denote the closed curve consisting of C_1 & $-C_2$ then by Cauchy's theorem



$$\int_C f(z)\,dz = 0$$

or, $$\int_{C_1} f(z)\,dz + \int_{-C_2} f(z)\,dz = 0$$

or, $$\int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz = 0$$

$$\therefore \int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz.$$

Cor. 2: Extensions of the Cauchy-Goursat theorem to multiply connected regions.

Let D be a doubly connected region

...unded by two simple closed curve C_1 & C_2 such that C_1 is contained in C then

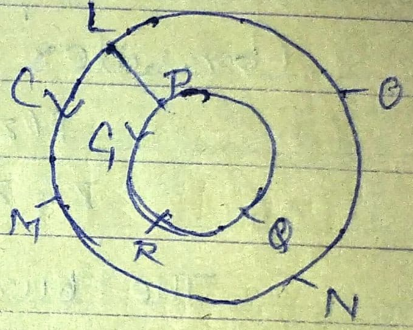
$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz$$

where, C & C_1 are both traversed in the +ve sense.

Proof:- We make a cross-cut joining a point L & C to a point P of C_1 the region from by L, M, N, O, L, P, Q, R, P, L is simply connected.

Hence, by Cauchy Goursat's theorem,

$$\int_{LMNOLPQRPL} f(z)\,dz = 0$$



$$\therefore \int_{LMNOL} f(z)\,dz + \int_{LP} f(z)\,dz + \int_{PQRP} f(z)\,dz$$

$$+ \int_{PL} f(z)\,dz = 0. \quad \text{---} (1)$$

$$\text{But} \int_{LP} f(z)\,dz = -\int_{PL} f(z)\,dz.$$

Hence, (1) reduces to

$$\int_{LMNOL} f(z)\,dz + \int_{PQRP} f(z)\,dz = 0$$

$$\text{or,} \int_{C} f(z)\,dz + \int_{-C_1} f(z)\,dz = 0$$

$$\therefore \int_{C} f(z)\,dz = \int_{C_1} f(z)\,dz.$$